A pseudo Circle expressed by Iterations: Stationary point, Symmetry, Fractal, Morphing

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Abstract

This is a text for Science Seminar 2018 in University of Edogawa. The object is to search interesting phenomena on computing. It's constructed as a reference material for discussions between a tutor and students.

Continuous movements in consecutive time are replaced by the iterative operations; and rendering patterns in the finite times are discussed. We show a figure like the circle, but the characters are different from it. After discussing phenomena, we modify surroundings of generating equations.

Keywords: complex iteration, finite circle, stationary points, backward operation, fractal, morphing, extended addition

1. Preface

In case of drawing a circle, we determine a center and draw an equidistant curve from there. The center and circumference is an inseparable pair. If a pattern like the circle is given, can the center be determined as one point, without using property of the circle?

We live in 3D space and 1D forward time. They are continuous (continuum cardinality) [1]. The meaning is learned in a general culture course. Any movement is done in the 4D. Mosaicking with fragments of movement, a pattern is generated, which can be done in the computer. The computer is a finite digital machine but not continuous. Thus; we wish to reduce above 4D time-space. At least, 1D time must be finite.

We adopt *the iteration* as a finite movement. It has not meaning of the time, but has integer suffix that is corresponded with the number of operations. It is acceptable to research patterns by using the computer.

2. Iteration generating a pseudo circle

Because of discussions for the circle, we set the space is 2D, and consider simple forward iteration first. The terminal condition is the number of operations. The descriptors are the complex number and 4 arithmetic operations and their functions.

The product of complex numbers is sum as for the arguments (θ), and the absolute is product of each absolute. Therefore; A first order iteration, "Z(n+1)=AZ(n), A=cos(θ)+*i*sin(θ), θ=2π/n, Z(0)=(1,0)", gives a circle of radius 1. Where, the center (0,0) is defined a priori. This is the same of traditional drawing.

We leave from the rendering, and don't use the center, and select iterations for gathering fragments of patterns. At least the quadratic form is required for rotation, translation, and extension/compression. We use,

$$Z(n+1)=AZ(n)^2+BZ(n)+C,\ 0\le n,\ |Z(n)|\le 1. \tag{1}$$

Where, Z(0), A~C are the complex, which are given initially. The "n" is a suffix. A condition of |Z(n)| is introduced to stop divergence. If researching diffusion property, the conditions is set as $\sim 10^{12}$. Eq.(1) gives a pattern like the circle under the following values.

$$Z(0)=(5\times10^{-4},\ 5\times10^{-4}),\ A=(1/2^{0.5},1/2^{0.5}),\ B=(0,-1),\ C=(-1\times10^{-4},\ 0),\ n\le 2\sim5\times10^{4}. \tag{2}$$

Calculations of Eq.(1) cannot be executed always because of the limitation of registers in a computer. Therefore, a termination condition, |Z(i)| < threshold, is required on programming.

A calculated pattern is in **Figure 1**; however, is it a circle truly? Let's test it.

Horizontal and vertical lines are the real and imaginary axes. Tiny dots are plotted per 1×10^{-4}. The plotting number is 5×10^{4}. "G" is a center of gravity point, (-4.9996×10^{-5}, 5.0003×10^{-5}). A mean point of the maximum and minimum points of the set is (-5.0000×10^{-5}, 5.035×10^{-5}). The difference is O(-7), where O(-7) means 10^{-7}. If it is a true circle, the both points are overlapped. Therefore; it is not the circle; but the difference is very small. Please remember the difference.

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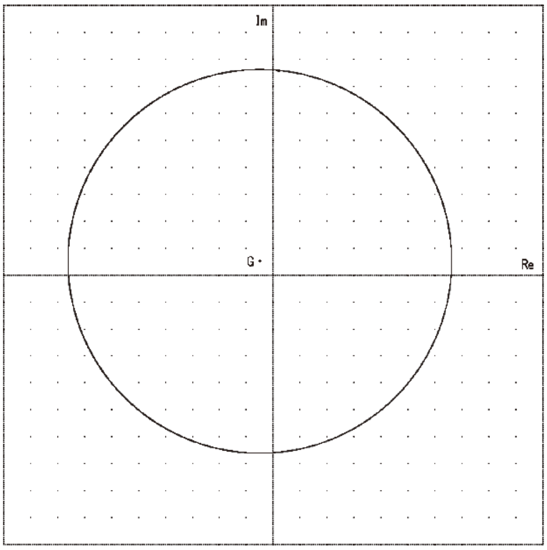

Figure 1. A set plotted by Eq.(1) and initial condition (2).

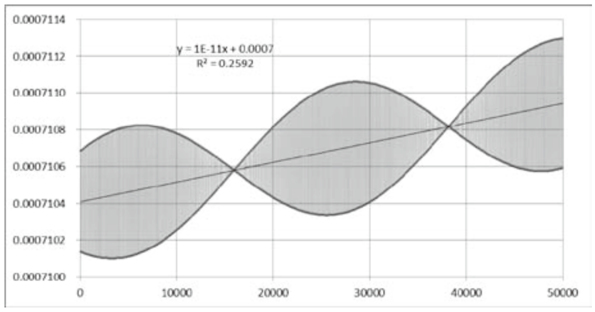

Figure 2. Distances between the G-point and circumference.

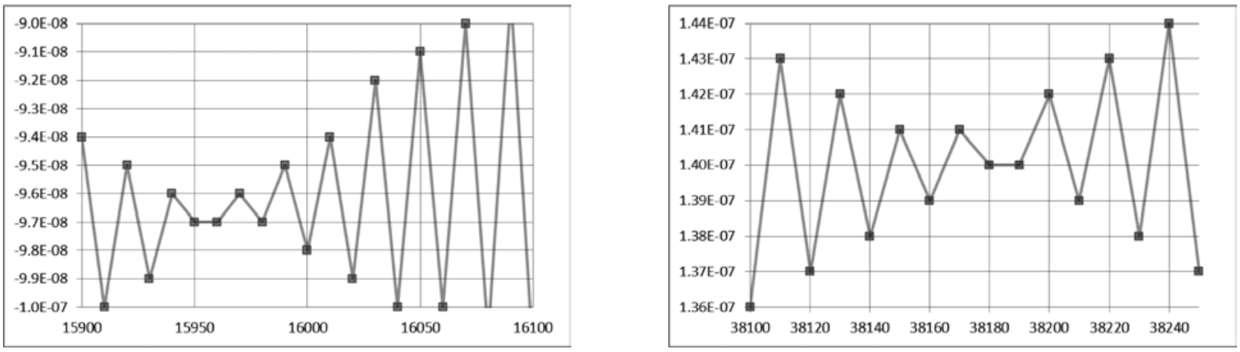

Figure 2A. Change of the difference between distances and the average one.

G-point is as if it is a center of the circle. The figure is 721×721 pixels (Below, the resolution is the same); thus O(-3) is less under 1-pixel. We call the **Figure 1** "pseudo circle (p-circle)". Distances between the G-point and points around the p-circle are not fixed, which is shown in **Figure 2**.

Distances have changed in $\Delta=10^{-7}$, and average is 7.10676×10^{-4}. Two waves are found, and both periods are same, and $\sim2.2\times10^{4}$ operations. Both distances increase linearly; thus, it is overlapped by 2 kinds of figures, and they would be divergence for the infinity of "n".

3. Stationary condition

A condition, Z(n)=Z(n+1), gives an equation and the roots,

$$AZ(n)^2+(B-1)Z(n)+C=0, \tag{3}$$

$$Z(n)=\{-b\pm(b^2-4AC)^{\underline{0.5}}\}/2A,\ b=B-1. \tag{4}$$

The 0.5 is the complex-root. The solution is, P0=(1.41426356237, -5×10^{-5}), P1=(-5×10^{-5}, 5×10^{-5}). If Z(0)=P1, Z(0)=Z(1)=…Z(∞)=P1; this is a stationary point; and it is almost the center of gravity point. The difference is O(-8).

But; the P0 gives divergence series. The cause is calculations on the computer are an approximation, they have small error. Therefore around unstable points, even if it is a root, the error puts to the divergent. **Figure 3** shows the accumulation of error.

Horizontal is the number of iteration. Vertical is the distance, which is between each iterative point and the initial point. Eq.(3) is kept until n~16.

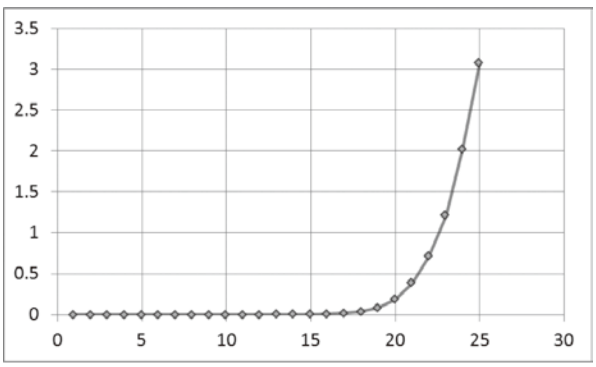

Figure 3. Distance divergence around an unstable root.

How do initial Z(0)'s around the unstable P0-point diverge?

$Z(0)kL=P0+(5k/8,\ 5L/8)\times 10^{-8},\ -8\le k\le 8,\ -8\le L\le 8,$ (5)

Where "k,L" is suffix, integer. Do they go to the infinity homogeneously? It's an interesting problem. To trace the divergence, a modified logarithm/square-root function is introduced.

mLog10(x)={Log10(x), 0<x; -Log10(-x), x<0}, (6A)

mSqrt(x)={sqrt(x), 0<x; -sqrt(-x), x<0}, (6B)

By using mLog()-function, we draw the divergence-trace in **Figure 4**.

The plane size is $[-10^{10},10^{10}]$ for real (Re)- and Imaginary (Im)-axes. The center is the origin, (0,0). 946k points are plotted. It's negligible small displacements of O(-8), but they cause the divergence.

What traces are there around P1-point? All are concentric circles.

The plane size is $[-10^{-3},10^{-3}]$ for Re- and Im-axes. The center is the origin, (0,0). Circles do not close because the number of iteration is small. The displacement is 5×10^{-5}, which is 103 times larger than that of P0-case.

4. Association sequence

An association sequence, As(), for Eq.(1) is defined.

As(k; n)=mod(n,k), 0≤n, 2≤k (integer). (7)

Using the As(), **Figure 1** is separated into 2 p-circles. Each p-circles have 2 G-point, $(-5.0009\times 10^{-5},\ 5.003\times 10^{-5})$, $(-4.998\times 10\text{-}5,\ 4.998\times 10^{-5})$. Difference of 2 G-points is O(-8).

Using the As (), we trace the situations at the start of plotting p-circles. We get **Figure 5**. The figure is not 2 p-circles, but is 4 p-circles. However, in the condition, n<200k, the gravities are changed as **Table 1**. It indicates each p-circle is not closed even if it is n~200k iterations.

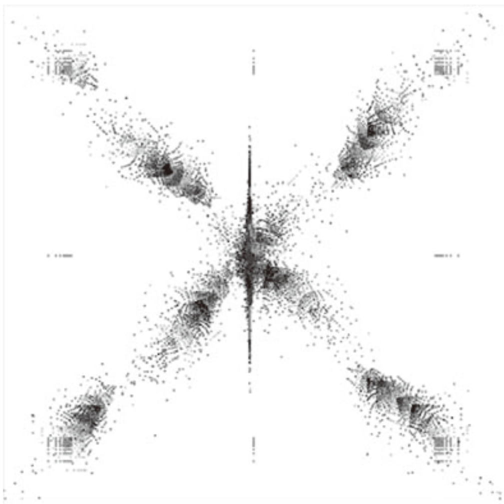

Figure 4. Divergence trace of 289 Z(0)'s around P0-point.

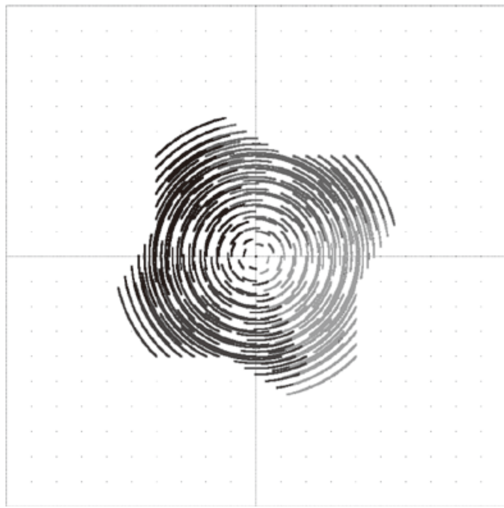

Figure 4B. Traces of 289 Z(0)'s around P1-point.

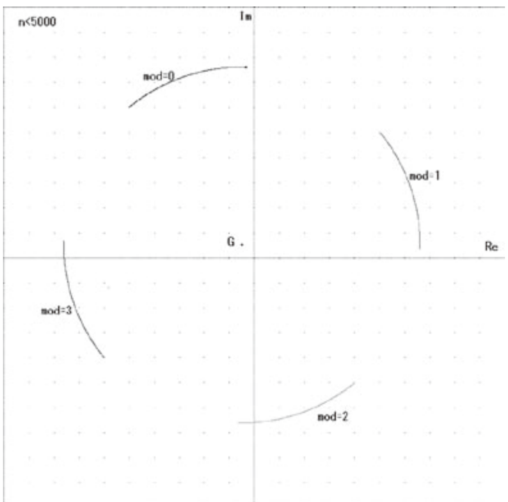

Figure 5. A set plotted by Eq.(1), initial condition (2), and n<5k. Where As(4; n) function is used.

Table 1. Center of gravity coordinates of 4 p-circles.

n<100k	X*	Y*
mod0	-0.428095	1.20913
mod1	0.208902	0.428164
mod2	-0.571602	-0.208731
mod3	-1.20898	0.571471
Average	-0.49994	0.500009
Gravity**	-0.49995	0.50001
n<200k	X	Y
mod0	-0.109245	0.816859
mod1	-0.182939	0.109343
mod2	-0.890667	0.182959
mod3	-0.816949	0.890676
Average	-0.49995	0.499959
Gravity**	-0.49995	0.499959

* unit [×10-4], ** Center of gravity for whole set.

We can make plotting be color by using the As(). However, under restriction of printing cost, the paper is monochrome.

5. Even-Odd expression

We discuss another expression of Eq.(1), $Z(n+1)=AZ(n)^2+BZ(n)+C$. Let $n\to n+1$, then we get,

$$Z(n+2)=AZ(n+1)^2+BZ(n+1)+C. \quad (8)$$

We substitute Eq.(1) into Eq.(8).

$$Z(n+2)=A^3Z(n)^4+2A^2BZ(n)^3+A\{B^2+2AC+B\}Z(n)^2 +\{2ABC+B^2\}Z(n)+C\{AC+B+1\}. \quad (9)$$

The Eq.(9) generates $\{Z(i), i=0,2,4,\ldots\}$, where there is no "i=odd". The trace is a p-circle. If Z(1) is got from Eq.(1), series $\{Z(i), i=1,3,5,\ldots\}$ is generated. Both series are p-circles; initials are different; *i.e.*, the phase.

Distances between the G-point $(-5\times10^{-5}, 5\times10^{-5})$ and circumference is in **Figure 6**. Where is difference from **Figure 2**? Eq.(9) arrives at Z(N) by 1/2 operations. Even though it is mathematics, the velocity and phase is, as if it is physics.

Next, we search the stationary points of Eq.(9). The condition is, $Z(n+2)=Z(n)$. (10)

Therefore; $0=A^3Z(n)^4+2A^2BZ(n)^3+A\{B^2+2AC+B\}Z(n)^2 +\{2ABC+B^2-1\}Z(n)+C\{AC+B+1\}$. (11)

Such a troublesome expression, we replace the coefficients.

$$0=Z(n)^4+PZ(n)^3+QZ(n)^2+RZ(n)+S, \quad (12A)$$

$$P=2A^{-1}B,\ Q=A^{-2}\{B^2+2AC+B\},$$
$$R=A^{-3}\{2ABC+B^2-1\},\ S=A^{-3}C\{AC+B+1\}. \quad (12B)$$

Where, the P~S is known. The Eq.(12A) has the roots of Eq.(1), which are α and β, and are known. Using $Z(n)\equiv Z$, we get;

$$0=Z^4+PZ^3+QZ^2+RZ+S=(Z-\alpha)(Z-\beta)(Z^2+UZ+V)$$
$$=\{Z^2-(\alpha+\beta)Z+\alpha\beta\}(Z^2+UZ+V)=(Z^2+GZ+H)(Z^2+UZ+V). \quad (13A)$$

$$G=-(\alpha+\beta),\ H=\alpha\beta, \quad (13B)$$

The G and H are known. Eq.(13A)$=Z^4+(U+G)Z^3 +(V+GU+H)Z^2+(GV+HU)Z+HV$, (13C) Thus; we get,

$$U+G=P,\ V+GU+H=Q,\ GV+HU=R,\ HV=S, \quad (14A)$$

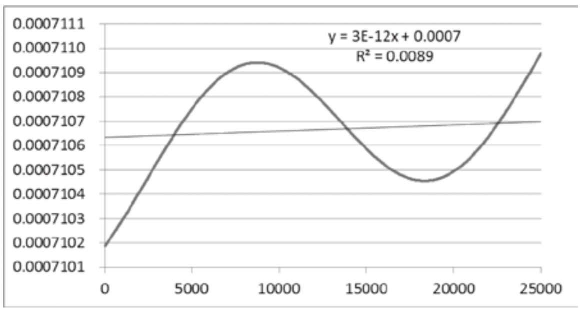

Figure 6. Distances between the G-point and circumference.

$$U=P-G,\ V=S/H. \quad (14B)$$

The U and V are determined. Thus, 4 roots of Eq.(9) are got. Using two equations, V+GU+H=Q, GV+HU=R, we can calculate test for the precision. We get V+GU+H-Q~O(-12) and GV+HU-R~O(-12).

By using case {A, B, C} of Eq.(2), we get 4 roots,

$P0=(1.41426356237, -5\times10^{-5})$, $P1=(-5\times10^{-5}, 5\times10^{-5})$,
$P2=(0.899464585372, 1.26295372924)$,
$P3=(-0.899464585372, 0.151259833133)$. (15)

Initial Z(0)s around P2 and P3 points give divergence traces (in **Figure 7A,B**). The P1 is a stable pole, and is the center of gravity.

The plane size is $[-10^{10}, 10^{10}]$ for Re- and Im-axes. The center is the origin, (0,0). It's negligible small displacements of O(-8), but they cause the divergence.

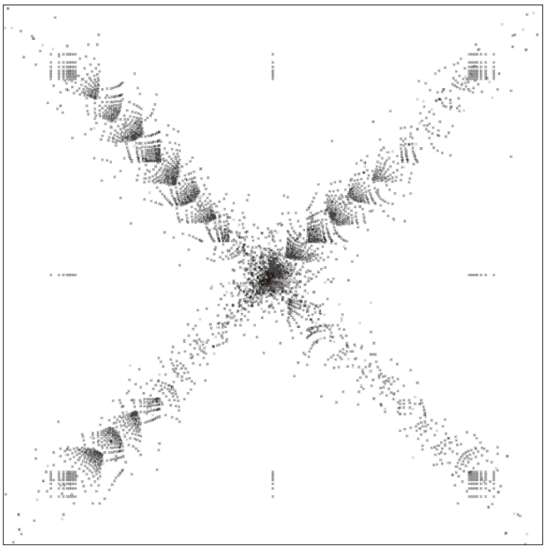

Figure 7A. Traces around P2 point.

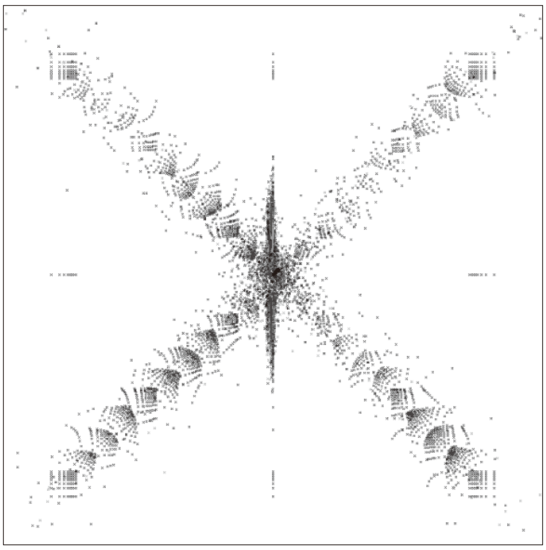

Figure 7B. Traces around P3 point.

6. Accumulation of shift registers

The shift registers are known, whose function is, Z(n)→Z(n+1)→Z(n+2). Modifying the sequence, and adding coefficients, we get;

$$Z(n+2)=AZ(n+1)^2+BZ(n)+C, \quad (16A)$$

$$Z(1)=AZ(0)^2+BZ(0)+C. \quad (16B)$$

It generates another series for Eq.(1). It generates vortices usually. However, introducing the stationary condition, Lim(n→∞): Z(n+2)=Z(n+1)=Z(n), the roots are, $0=AZ^2+(B-1)Z+C\equiv$ Eq.(3). Same {A, B, C} is adopted, the roots are that of Eq.(3). Practically, the P1 is stable pole, and the P0 is unstable. {Z(i)} is in **Figure 8**.

The plane size is $[-2.5\times10^{-1}, 2.5\times10^{-1}]$. Z(0) is the P1-point. The center of vortex-part is the G-point. This is regression curves, which are detected by **Figure 9**. The period is swayed.

Vertical (logarithmic scale) and horizontal axes are the distance and the last suffix number, whose unit is 1k. The distance is |Z(i)-G|.

The suffix value is blocked every 50k iterations. The maximum/minimum distances are calculated per a block. The minimum distance is O(-12); this is reasonable on the precision of calculations. It is interesting such phenomenon is found around a stable root.

7. Backward iteration

A backward iteration can be defined

$$0=AZ(n)^2+BZ(n)+C-Z(n+1),\ 0\le n, \quad (17)$$

It is not forward direction of Eq.(1) but backward from a constant Z(n+1)=Z(L); L=large number. Since root numbers of Eq.(17) are infinity, we introduce a restriction. Even in such a simple generation system, there are multiple locations generating one point.

$$0=AZ(n)^2+BZ(n)+\{C-Z(n+1)\}, \quad (18A)$$

$$Z'(n)=\min\{|Z(n,0)-P|,|Z(n,1)-P|\}. \quad (18B)$$

$$Z'(n)=\max\{|Z(n,0)-P|,|Z(n,1)-P|\}. \quad (18C)$$

Where, Z(n,0) and Z(n,1) are the root of Eq.(18A).

By using the restriction Eq.(18B), one Z'(n) is determined, and we rewrite Z(n). The "P" is P0 or P1 of Eq.(4).

As an example, $Z(L)=(5\times10^{-5}, -5\times10^{-5})$, it is the symmetric point of P1 and (0,0). After n=50k and P≡P1, we get a p-circle, and by using P≡P0 (unstable), we get a convergence trace onto P0. On other hand; in case of Eq.(18C), we get a p-circle (G-point is P1) by using P≡P0. When P≡P1 is used, a convergence trace onto P0 is got. The backward iteration suggests two roots {P0,P1} is related.

Above underlined fact indicates that a p-circle can be started from a target point (the gravity is P1-point). Since the p-circle is constructed with finite points, there are gaps along the circumference. To fill the gap, we can draw a p-circle from an intermediate location among the gap's point. Does the p-circle overlap with original one?

Using parameters of Eq.(2) (n≤20k), an intermediate point is calculated from Eq.(19).

$$Z(L)=(Z(U)+Z(V)/2. \quad (19)$$

Where, U=10k, V=10k+4. The "U" is given, and the "V" is got by searching the minimum distance in {Z(i), i≤20k}. The results are; $Z(U)=(4.23063\times10^{-4},5.80490\times10^{-4})$, $Z(V)=(4.23364\times10^{-4},5.80222\times10^{-4})$, $Z(L)=(4.23213\times10^{-4},5.80356\times10^{-4})$. The gravities of 2 p-circles are; $G(\text{Eq.(2) n=20k})=(-4.99505\times10^{-5},4.99993\times10^{-5})$,

$G(\text{new p-circle})=(-4.99951\times10^{-5},5.00458\times10^{-5})$.

The mean of distance among points on 2 circumferences is 1.848×10^{-7} that is less 1-pixel. Since such a new p-circle can be drawn, the p-circle and lines intersect within O(-7).

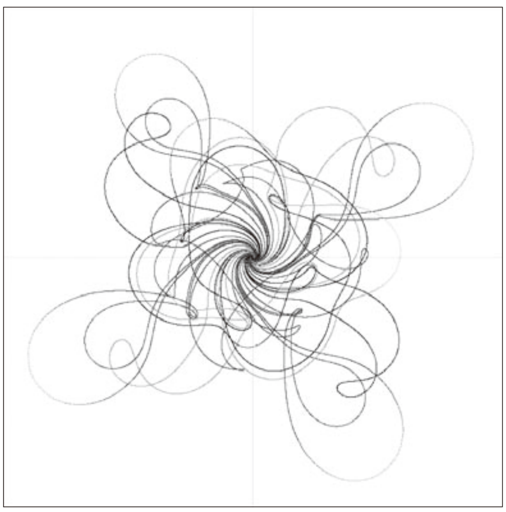

Figure 8. Plotting of Eq.(16A, B) until n=2M.

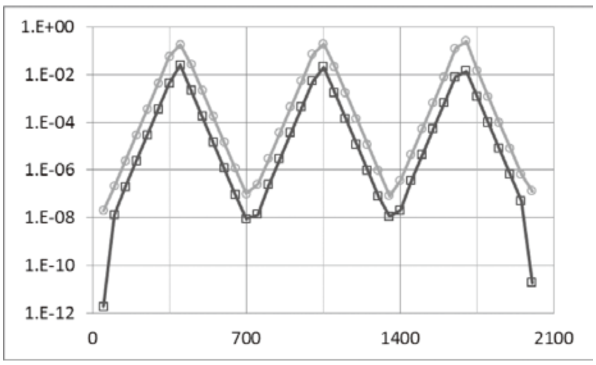

Figure 9. Change of distances per 50k iterations.

8. Quadruple expression

We make renormalization for Eq.(9), and get;

$$Z(n+2)=A^3Z(n)^4+PZ(n)^3+QZ(n)^2+RZ(n)+S, \quad (9a)$$

$$P=2A^2B,\ Q=A\{B^2+2AC+B\},$$
$$R=\{2ABC+B^2\},\ S=C\{AC+B+1\}, \quad (9A)$$

$$Z(n+4)=A^3T^4+PT^3+QT^2+RT+S \equiv \textit{function of } Z(n),$$
$$T=A^3Z(n)^4+PZ(n)^3+QZ(n)^2+RZ(n)+S. \quad (9B)$$

Where; P~S are working variables, at using here only. By using the quadruple expression, we can do following calculations. $Z(n)\rightarrow Z(n+4)\rightarrow \underline{Z(n+3)\rightarrow Z(n+2)\rightarrow Z(n+1)}$; $Z(n+8)\rightarrow\ldots$ (20)

The underlined part is the backward iteration. From one Z(n), 8 Z(n+1)s are calculated; thus, the number of Z(n)~Z(n+4) is 16. It is certain they are on one circumference; we confirm the fact. It is something strange if a different pattern is generated just by changing the calculation order. Repetition of finite operations can be partly reversed. In other words, there is a part that can go backward during forward operation. But the result is the same.

9. Symmetry

The calculation precision is checked in section 5, which is O(-12). The G-point is (-4.9996×10^{-5}, 5.0003×10^{-5}), and the mean point is (-5.0000×10^{-5}, 5.035×10^{-5}) in section 2. The difference is very small, but it is signify. We count the number of Z(i) in each orthant.

Considering **Table 2**, the mean-point is not appropriate as the center. We should adopt the G-point for the center. Distribution of points has a symmetry whose axis is a vertical line on G-point. There is no symmetry for a horizontal line; therefore, it is almost C2-symmetry (or mirror image). The iteration is finite plotting; then, words of the point group are not used. However, since the difference is O(-8) and they are well known, here, we use them.

Using a point Z(i) in orthant(I) and the C2-symmetric "C2(Z(i))", the minimum distance point Z(Jmin) is defined.

Table 2. The number of Z(i) in the orthant.

G*	I**	II	III	IV
n<50k	12497	12496	12504	12503
n<200k	49985	49985	50015	50015
Mean*	I	II	III	IV
n<50k	12492	12491	12509	12508
n<200k	49969	49967	50032	50032

* Location of the center. ** The number of orthant.

$$C2(Z(i))=2Gx-Z(i),$$
$$Z(Jmin)=\min\{|C2(Z(i))-Z(j)|\},\ Z(j)\subset \text{orthant(II)}. \quad (21)$$

Distribution of the Z(Jmin) is in **Figure 10**.

Vertical axis is the distance, horizontal is the number of iterations. For each iteration, the minimum point exists homogeneously. However, the minimum value is not same. Moreover, it is never zero. Therefore, operations of the point group are not consistent. This is different from the circle. The position of small values ($<5\times10^{-8}$) is in **Figure 11**.

Vertical and horizontal axes are Imaginary- and Real-axis. Orthant(II) part of p-circle is drawn, the bold curves are $\{Z(Jmin)\}<5\times10^{-8}$ of Eq.(20). That is; the bold part has C2-symmetry closely. The circumference of a p-circle is not uniform on the view point of symmetry.

10. Fractal character

Until now it is a discussion of finite point set in the complex plane. From here on; we introduce the continuous cardinality [1], in which polygons are researched.

We define a circumference (CR) based on the series {Z(i)}.

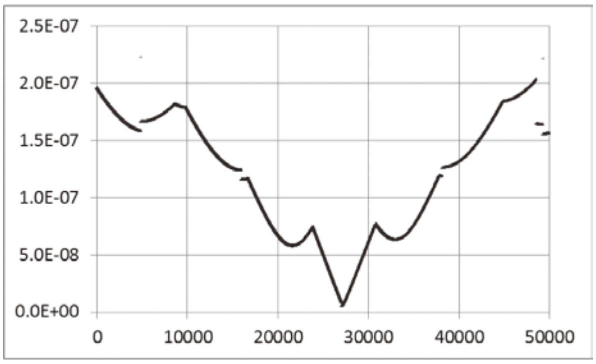

Figure 10. Distribution of the Z(Jmin) in 50k iterations.

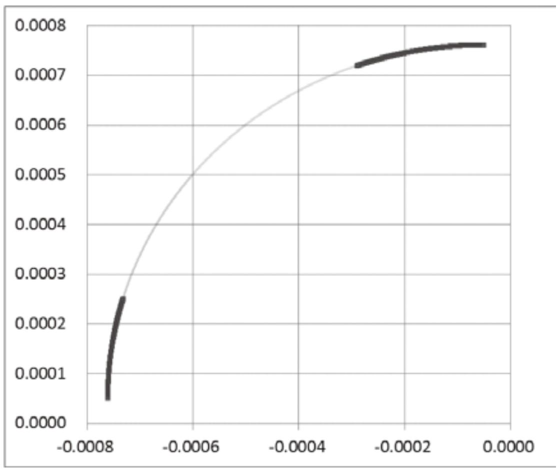

Figure 11. Locations of Z(Jmin) in orthant(II).

Start: i=1, $Z(j)=\min\{|Z(j)-Z(i)|,\ 1\le j\le n,\ j\ne i\}$, (21A)

$Z(k)=\min\{|Z(k)-Z(j)|,\ 1\le k\le n,\ k\ne j,\ k\ne i\}$, (21B)

$Z(l)=\min\{|Z(l)-Z(k)|,\ 1\le l\le n,\ l\ne k,\ l\ne j,\ l\ne i\}$, (21C)

…..

$CR=\{Z(i), Z(j), Z(k), Z(l),\ldots.\}$. (21D)

There is a line between adjacent pair $\{Z(i),Z(j)\}$,… $\{Z(last),Z(i)\}$. Eq.(21D) determines a closed curve. The length (L) of a circumference is,

$L=\Sigma_{i,j,k,\ldots}|Z(i)-Z(j)|+|Z(j)-Z(k)|+|Z(k)-\ldots|+\ldots$. (21E)

The area (S) is by using Heron's formula [2],

$S=\Sigma_{i,j,k,\ldots}H\{Z(i),Z(j),G\}+\ldots$.,

$H(x,y,z)=\{(s-x)(s-y)(s-z)\}^{0.5}$, $s=(x+y+z)/2$. (21F)

Horizontal axis is the number of iterations (n), unit [k] operations. Left vertical axis is scale for radius and area; and the right is for the circumference plots.

The circumference and area have the powers of 0.709 (R^2=0.97), 0.608 (R^2=0.95), respectively; where increasing effects of the radius are corrected. Since the true circles have both zeroes, the p-circle has fractal character [3]. It is arisen by invisible zigzag on the circumference. The zigzag has radial/tangential components, which give a difference of circumference/area. This is not the Koch curve, but is an interesting property.

11. Extension to quadratic curves

Hereafter, it is countable cardinality space again.

11.1 Ellipse

We set the origin iteration to Eq.(16A), but Z(1) is replaced by Eq.(16C). Z(0) and A~C parameters are Eq.(16D).

$Z(n+2)=AZ(n+1)^2+BZ(n)+C$, (16A)

$Z(1)=\mathrm{conjg}\{Z(0)\}$. (16C)

$Z(0)=(5\times10^{-4}, 0)$, $A=(1/2^{0.5},1/2^{0.5})$,

$B=(0, -1)$, $C=(1\times10^{-4}, 1\times10^{-4})$. (16D)

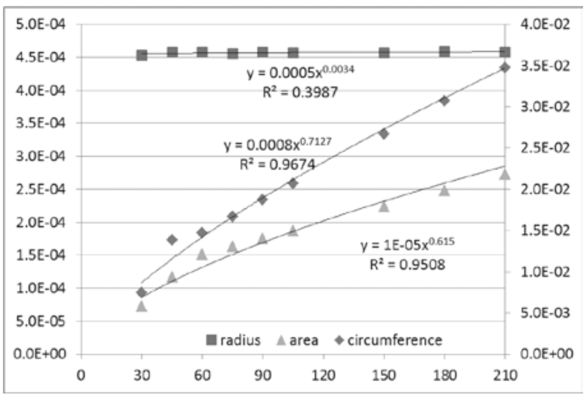

Figure 12. Fractal character of p-circle for the number of iterations.

The different from Eq.(2) is Z(1) and C terms. Moreover, the condition of Eq.(16E) is required. The As() is *cf.*Eq.(7).

IF{As(4; n)=L}=true, 33k < n. (16E)

Where "L" is {0,1,2,3}. If Eq.(16E) is erased, 4 ellipses are drawn (**Figure 13A**).

11.2 Hyperbola

Using Eq.(16A,C), a pattern like the hyperbola (**Figure 13B**) is drawn under Eq.(16F). Eq.(16E) is not used.

$Z(0)=(\underline{1\times10^{-4}}, 0)$, $\underline{n\le 90k}$, $A=(1/2^{0.5},1/2^{0.5})$,

$B=(-1, 0)$, $C=(-1\times10^{-4},-1\times10^{-4})$. (16F)

The underlined conditions are important; if not, regression curves (*cf.* **Figure 13C**) are drawn; this is not hyperbola but is a pseudo hyperbola stopping at finite points.

The sizes are $[-1\times10^{-3}, 1\times10^{-3}]$. G is the center of the gravity. Actually, the pseudo hyperbola is also made up of 4 parts. You can understand it by coloring with As (4,)-function.

The plane sizes are $[-3\times10^{-2}, 3\times10^{-2}]$. A~C parameters are in Eq.(16F). **Figure 13B** is small area around the origin of **Figure 13C**. The shape is nearly hyperbola; however it is kept nearby the origin and under small "n" only.

11.3 Fractal dimension of Arc length and Area

Using Eqs.(16A,C,D,E), we calculate the arc length and area of an ellipse. In case of n={32k, 48k, ….} and under As(1,)-function, the effective plot number is Eff(n)={8k, 16k, …, 56k}. We get **Figure 13D**.

Horizontal is the number of iterations [k], the left vertical is for the area, and the right one is for the arc length. The dependence values are 1.6×10^{-4} (R^2=0.98), 4.9×10^{-5} (R^2=0.96) for the area and arc length. Both are nearly zeroes; no Fractal character is found.

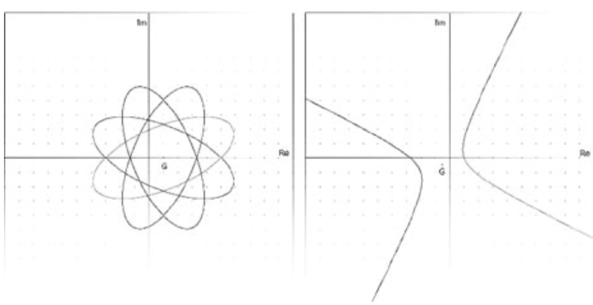

Figure 13A. 4 Ellipses. Figure 13B. pseudo Hyperbola.

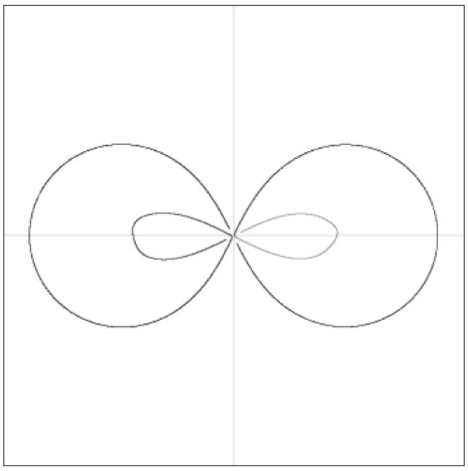

Figure 13C. Regression curves under "n=120k".

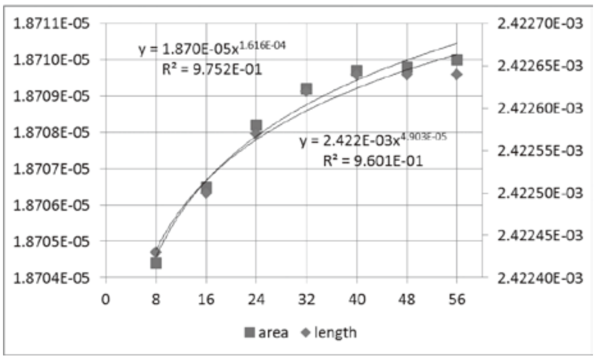

Figure 13D. Number dependency for an iterative ellipse.

12. Morphing

12.1 From ellipse to hyperbola

On rendering of section 11, it is data of B and C, not Eq.(16A) that determines the shape. What is the shape if it is intermediate values between B and C? Have the shapes a remaining of the ellipse or hyperbola? It is called Mathematical Morphology (MM) [4]. It writes that topological and geometrical continuous-space concepts such as size, shape, convexity, connectivity, and geodesic distance, are introduced by MM on both continuous and discrete spaces.

A real parameter "ω" is defined to control shape ratio.

$0 \leq \omega \leq 1$, $\sim\omega = 1 - \omega$, $Bm = \sim\omega Be + \omega Bh$, $Cm = \sim\omega Ce + \omega Ch$. (22)

The Be, Ce, Bh, Ch are the B and C for the ellipse and hyperbola, respectively. The maximum "n" of Eq.(16A) is 300k. We adopt a series, $\omega = \{10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}\}$.

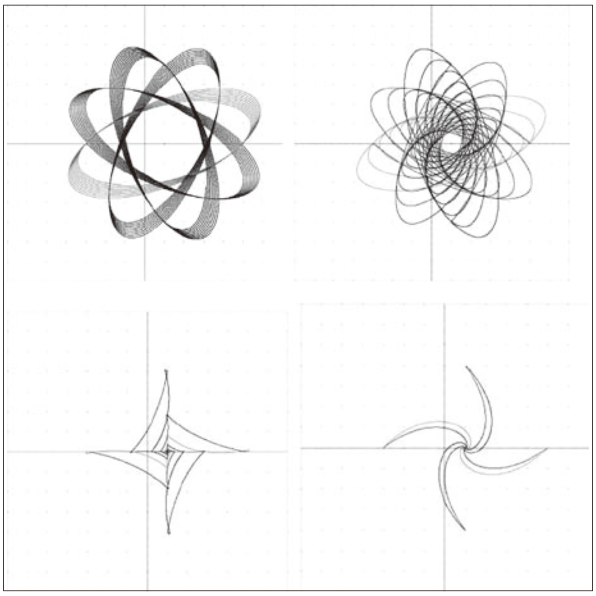

Figure 14. Morphing patterns for small displacements from the ellipse.

The plane size is $[-7\times10^{-4}, 7\times10^{-4}]$. The pitch of small dots is 1×10^{-4}, Horizontal and vertical axes are the real- and Imaginary. The ω-series' order is "from the top left to right; one step down, left, then right". The top left is $\omega=10^{-6}$. The bottom right is 10^{-3}. The movements of the center of gravity are in **Table 3**. All patterns can make coloring by As(4,)-function.

Table 3. Locations of Gravity for ~ω-parameter.

Value of ω	Location of G
1×10^{-6}	$(1.00008\times10^{-4}, -3.6\times10^{-10})$
1×10^{-5}	$(1.00005\times10^{-4}, -4.8\times10^{-10})$
1×10^{-4}	$(9.999\times10^{-5}, 8.7\times10^{-9})$
1×10^{-3}	$(9.98\times10^{-5}, 9.8\times10^{-8})$

12.2 From pseudo hyperbola to ellipse

Figure 15. Morphing patterns for small displacements from pseudo hyperbola around the origin.

The plane sizes are $[-2\times10^{-3}, 2\times10^{-3}]$, $[-1\times10^{-3}, 1\times10^{-3}]$ for the left and right. The pitch of small dots is 1×10^{-4}, Horizontal and vertical axes are the real- and Imaginary. The left is $\sim\omega=10^{-5}$, and right is 10^{-4}. The number of plotting is 90k. 2 patterns can make coloring by As(4,)-function. The G of $\sim\omega=10^{-4}$ is $(-4.9998\times10^{-5}, -4.998\times10^{-5})$.

13. In/outsides of a closed curve

By using Eqs.(16A,C,D,E), we calculate complex points, and connect the nearest contact ones. The connection is continuum cardinality. Thus; an ellipse is got. The A is,

$A = (\cos(\theta), \sin(\theta))$, $\theta = \pi/4 = 45$ [deg]. (16D')

The ellipse is a closed curve, which divide 2D-plane into in/outsides.

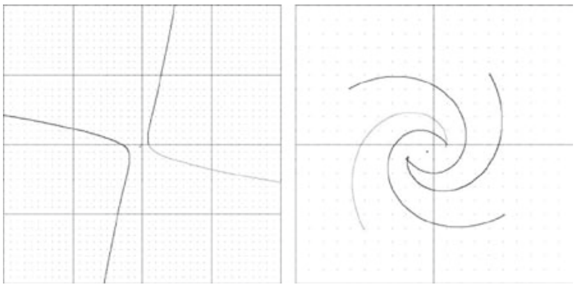

Figure 15

When the θ is set to be 15, 30, 60 [deg], all vortices are got. When θ=47.475 [deg], two vortex lines are overlapped, and **Figure 16** is got.

The plane size is [-7×10^{-4}, 7×10^{-4}]. The pitch of small dots is 1×10^{-4}, Horizontal and vertical axes are the real- and Imaginary. The number of plots is 75k; G=(1×10^{-4}, 0), As(4,)=1.

The vortex is constructed with 2 curves, and 2 terminal points, L0/L1. The 2 curves go to the infinity at "n→∞", and divide 2D-plane into A/B areas. The division is the same of that of the ellipse. It is reasonable such a shape is got by continuous morphing of the ellipse.

Since there are 2 lines; is the space divided into 3? We think that the terminals L0/L1 of 2 curves are connected somewhere at the infinity. If they don't join, they remain 2; A and B regions can not be distinguished. If you forcibly connect L0 and L1 in a finite world, the A and B areas become inside or outside of one curve. If there is a bifurcation somewhere in curve, the plane is divided into 3 areas. The restriction of Eqs.(21B,C) doesn't make such a branch.

14. Around the addition

Until now we pay attention to figures around points.

In other words, we watch the response to the change of data, $\partial f(x)/\partial x$. Let's change the point of view and consider the periphery of operations, which is a component of the equation. In the physics, the number represents the measured amount of something; Units are attached to numbers. If there are units in the number, addition and multiplication are heterogeneous. Even if an operation is the multiplication between different units, it can be executed and generate a new unit. On other hand, the addition cannot be executed. However; in the mathematics, addition and multiplication are homogeneous.

In the real numbers, X, Y, U, and V;

X+Y=Ln{exp(X)}+Ln{exp(Y)}

=Ln(U)+Ln(V)=Ln(U*V), Ln()=$\log_e$(). (23)

Eq.(23) is a correspondence relation (*→+).

Ln(U+V)=Ln{exp(X)+exp(Y)}=X(+)Y. (23A)

Eq.(23A) is a correspondence relation {+→(+)}.

Ln{U(+)V}=Ln[Ln{exp(exp(X))+exp(exp(Y))}]

=X$(+)_2$Y. (23B)

Eq.(23B) is a correspondence relation {(+)→$(+)_2$}.

Thus; we get a series of additions, {+, (+), $(+)_2$,…$(+)_n$}. We predict, Limn→∞[X$(+)_n$Y]=max(X,Y). (23C)

Hereafter, in the complex logarithm, the main value is assumed. Eq.(23A) is held on the complex number. The (+)-expression of Eq.(1) is,

Z(n+1)=AZ(n)^2 (+) BZ(n) (+) C, 0≤n. (24)

The parameters are,

A=(0.5,0.866025)={cos(60*),isin(60*)}, * [deg],

B=(0,-1), C=(-1×10^{-4},0), Z0=(5×10^{-4}, 5×10^{-4}). (24A)

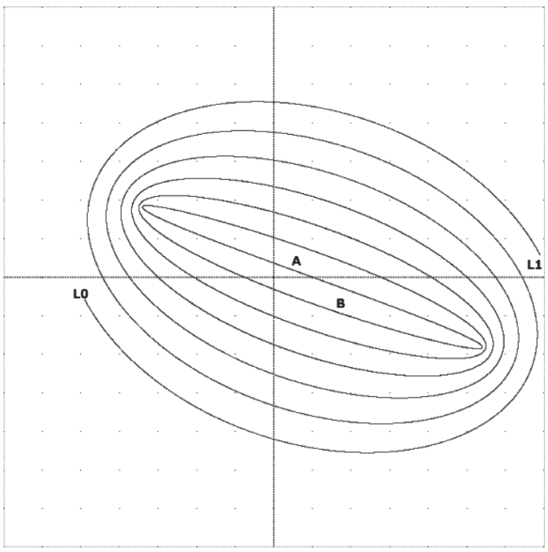

Figure 16. Vortex derived Eqs.(16A,C,D,D',E).

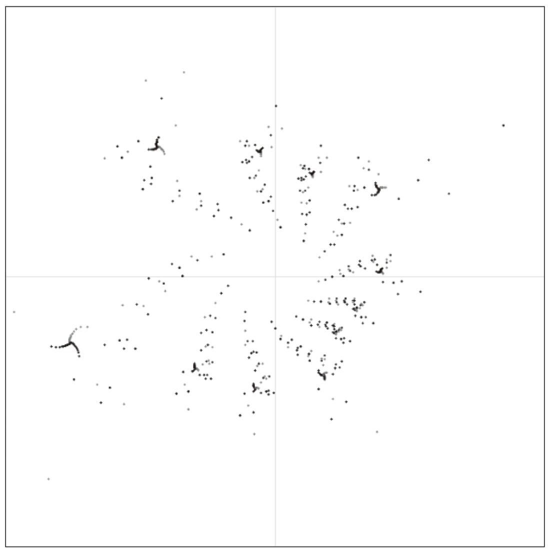

Figure 17. Dots assemble of Eq.(24, 24A).

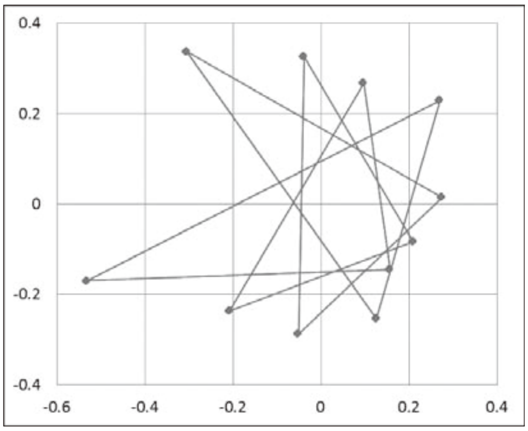

Figure 18. Location order* after 89k iterations, and the stationary points.

Table 4. Coordinates of stationary points.

St(#)	Count*	Re	Im
1	8141	1.21504	0.07515
2	8130	1.21063	0.28988
3	8142	1.03744	0.32880
4	8140	0.90218	0.38700
5	8125	0.63525	0.39876
6	8115	0.40894	-0.10939
7	8140	0.73300	-0.17652
8	8140	0.88946	-0.22701
9	8136	1.06626	-0.19307
10	8149	1.09730	-0.08414
11	8149	1.15020	-0.02244
total	89507**		

* The number of dots is in radius 0.005, the centers are stationary points. They are almost equal; thus, there is no accumulation point. The iteration can take such a state of circulating stationary points as well.

** All dots are 90k.

Table 5. Iteration sequences.

Eq.(1)	**z0**	z1						
		z1	z2					
			z2	z3				
				z3	z4			
Eq.(16A)	**z0**	**z1**	z2					
		z1	z2	z3				
			z2	z3	z4			
				z3	z4	z5		
Eq.(25A)	**z0**		**z2**	z3				
		z1		z3	z4			
			z2		z4	z5		
				z3		z5	z6	
					z4		z6	z7

Bold elements are initial values.

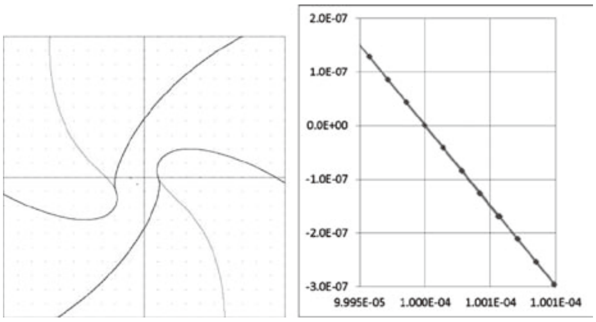

Figure 19. A set generated by Eqs.(25A~C); n=120k.

The plane size is [-0.7, 0.7]. Horizontal and vertical axes are the real- and Imaginary. The number of plots is 90k, G=(0.940506,0.060627), coloring is As(3,)-function. 11 stationary points are found.

The scale's values are from G-point. * They are St(1)→St(5)→St(9),→St(2)→St(6)→St(10), →St(3)→St(7)→St(11), →St(4)→St(8)→St(1). The lines are virtual, where continuous dots don't exist.

15. Branch

A modification of Eq.(16A) is,

$Z(n+3)=AZ(n+2)^2+BZ(n)+C,$ (25A)

$Z(1)=\mathrm{conjg}\{Z(0)\}, Z(2)=Z(1).$ (25B)

$Z(0)=(1\times10^{-4}, 0), A=(1/2^{0.5}, 1/2^{0.5}),$
$B=(-1, 0), C=(-1\times10^{-4}, -1\times10^{-4}).$ (25C)

The sequence is in **Table 5**.

The size is $[-1\times10^{-3}, 1\times10^{-3}]$. It is colored by As(3,). The right figure is drawn by iteration number, n=1~56 and around $Z(0)=(1\times10^{-4},0)$. The line is for understanding plot order. Only points are the real.

Points are real, and lines are to show the plotting order. This is a branch as the iteration, which is repeating among 3 points, and gradually moves to each direction.

Dot association around stationary points in section 14 is like a branch; however, the direction is convergence. Here branch is divergence.

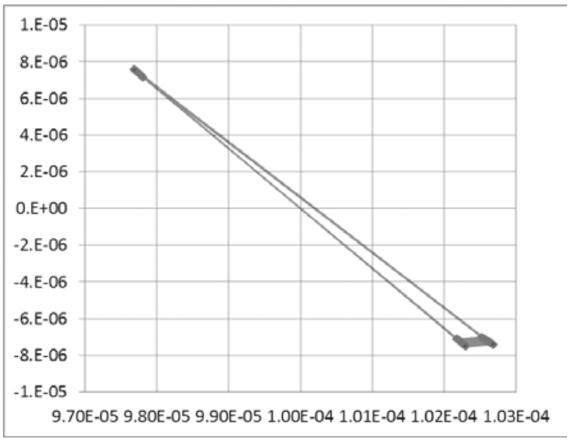

Figure 20. Dots between n=999~1064, around Z(0)=Z(1)=Z(2).

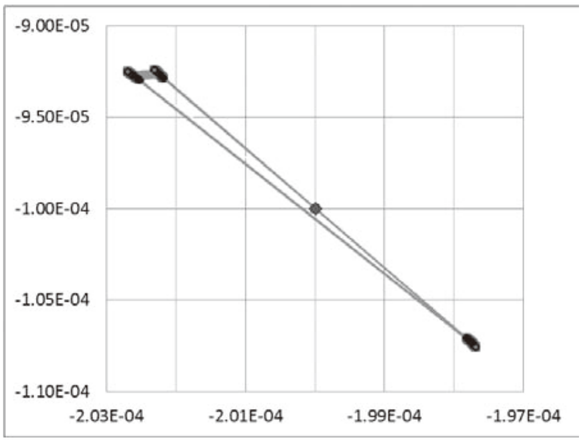

Figure 21. Dots between n=996~1067, around Z(3)=(-2×10^{-4}, -1×10^{-4})=Z(4)=Z(5). Location of Z(3) is added.

16. Conclusion

1) We discuss a method of drawing figures with operation counts. It is based on a thinking point somewhat closer to physics from mathematics. Introducing conception of iteration, under appropriate parameters, the iteration makes dots like the circle. But, they have characters different from the circle. Detecting the different, it's *an entrance of the science*.

2) When discussing figures of finite drawing, we need to think about *what will happen at the infinite*. Iterations can be set the stationary condition, which derives multi order equations. The roots are classified by surroundings stable/unstable property. There is interesting patterns around the stable root that is near the center of gravity.

3) Introducing shift registers into the multi order equations, we arrive at another iteration. It generates regression curves, in which vortices are found nearby the gravity point. Under trial parameterizations, pseudo quadric curves are found. Where, the determination of shapes is not expression but is parameter's values. On the point of view, we reach *conception of morphing*.

4) Other representation of iteration is a backward one. When multiple order terms are included in the expression, the backward derives *multi former locations for a current*. However, other shapes different from a forward iteration are not generated by the backward.

5) The symmetry of finite plotting patterns is considered. It is an approximation follows the point group. We detect the circumference is not uniform. By introducing lines, arising conception of inside/outside in the shape, the circumference and area have a *fractal character*.

6) We need to investigate around 4 arithmetic operations. This time, we examine the addition and apply it to the iteration that generates a pseudo circle. We find a set going around stationary points. Considering the branch pattern, when there are multiple equivalent stationary points and infinite operations; circulating motion among the stationary points is inevitable.

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